

SOLUTION OF GENERALIZED FRACTIONAL REACTION-DIFFUSION EQUATIONS

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Abstract. This paper deals with the investigation of a closed form solution of a generalized fractional reaction-diffusion equation. The solution of the proposed problem is developed in a compact form in terms of the H-function by the application of direct and inverse Laplace and Fourier transforms. Fractional order moments and the asymptotic expansion of the solution are also obtained.

1 Introduction

It is a known fact that reaction-diffusion models play a very important role in pattern formation in biology, chemistry, and physics, see Wilhelmsson and Lazzaro (2001) and Frank (2005). These systems indicate that diffusion can produce the spontaneous formation of spatio-temporal patterns. For details, one can refer to the work of Nicolis and Prigogine (1977) and Haken (2004). A general model for reaction-diffusion systems is investigated by Henry and Wearne (2000, 2002) and Henry et al. (2005).

The simplest reaction-diffusion models are of the form

$$\frac{\partial N}{\partial t} = d \frac{\partial^2 N}{\partial x^2} + F(N), N = N(x, t), \quad (1)$$

where d is the diffusion constant and $F(N)$ is a nonlinear function representing reaction kinetics. It is interesting to observe that for $F(N) = \gamma V(1 - N)$, eq.(1) reduces to Fisher-Kolmogorov equation and if we set $F(N) = \gamma N(1 - N^2)$, it gives rise to the real Ginsburg-Landau equation. Del-Castillo-Negrete, Carreras and Lynch (2002) studied the front propagation and segregation in a system of reaction-diffusion equations with cross-diffusion. Recently, del-Castillo-Negrete et al (2003) discussed the dynamics in reaction-diffusion systems with non-Gaussian diffusion caused by asymmetric Lévy flights and solved the following model

$$\frac{\partial N}{\partial t} = \eta D_x^\alpha N + F(N), N = N(x, t), \quad (2)$$

with $F = 0$.

In this paper, we present a solution of a more general model of fractional reaction-diffusion systems (2) in which $\frac{\partial N}{\partial t}$ has been replaced by the Riemann-Liouville fractional derivative ${}_0D_t^\beta, \beta > 0$. The results derived are of a more general nature and than those investigated earlier by many authors, notably by Jespersen, Metzler, and Fogedby (1999) for anomalous diffusion and del-Castillo-Negrete et al. (2004) for the reaction-diffusion systems with Lévy flights, and fractional diffusion equation by Kilbas et al. (2005). The solution has been developed in terms of the H-function in a compact and elegant form with the help of Laplace and Fourier transforms and their inverses. Most of the results obtained are in a form suitable for numerical computation. The present study is in continuation of our earlier works, Haubold (1998), Haubold and Mathai (2000), and Saxena, Mathai, and Haubold (2002, 2004a, 2004b, 2004c, 2005).

2 Mathematical Preliminaries

A generalization of the Mittag-Leffler function (Mittag-Leffler, 1903, 1905),

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + 1)}, \alpha \in C, Re(\alpha) > 0, \quad (3)$$

was introduced by Wiman (1905) in the generalized form

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + \beta)}, (\alpha, \beta \in c, Re(\alpha) > 0, Re(\beta) > 0). \quad (4)$$

The main results of these functions are available in the handbook of Erdélyi, Magnus, Oberhettinger, and Tricomi (1955, Section 18.1) and monographs by Dzherbashyan (1966, 1993). The H-function is defined by means of a Mellin-Barnes type integral in the following manner (Mathai and Saxena, 1978),

$$\begin{aligned} H_{p,q}^{m,n}(z) &= H_{p,q}^{m,n} \left[z \left| \begin{matrix} (a_p, A_p) \\ (b_q, B_q) \end{matrix} \right. \right] \\ &= H_{p,q}^{m,n} \left[z \left| \begin{matrix} (a_1, A_1), \dots, (a_p, A_p) \\ (b_1, B_1), \dots, (b_q, B_q) \end{matrix} \right. \right] = \frac{1}{2\pi i} \int_{\Omega} \Theta(\xi) z^{-\xi} d\xi, \end{aligned} \quad (5)$$

where, $i = (-1)^{1/2}$,

$$\theta(\xi) = \frac{[\prod_{j=1}^m \Gamma(b_j + B_j \xi)] [\prod_{j=1}^n \Gamma(1 - a_j - A_j \xi)]}{[\prod_{j=m+1}^q \Gamma(1 - b_j - B_j \xi)] [\prod_{j=n+1}^p \Gamma(a_j + A_j \xi)]}, \quad (6)$$

and an empty product is always interpreted as unity; $m, n, p, q \in N_0$ with $0 \leq n \leq p, 1 \leq m \leq q, A_j, B_j \in R_+, a_i, b_j \in R$ or $C (i = 1, \dots, p; j = 1, \dots, q)$ such that

$$A_i(b_j + k) \neq B_j(a_i - l - 1), k, l \in N_0; i = 1, \dots, n; j = 1, \dots, m, \quad (7)$$

where we employ the usual notations: $N_0 = (0, 1, 2, \dots)$; $R = (-\infty, \infty)$, $R_+ = (0, \infty)$ and C being the complex number field. The contour Ω is either $L_{-\infty}$, $L_{+\infty}$ or $L_{i\gamma\infty}$. The following are the definitions of these contours.

(i) $\Omega = L_{-\infty}$ is a left loop situated in a horizontal strip starting at the point $-\infty + i\varphi_1$ and terminating at the point $-\infty + i\varphi_2$ with $-\infty < \varphi_1 < \varphi_2 < +\infty$;

(ii) $\Omega = L_{+\infty}$ is a right loop situated in a horizontal strip starting at the point $+\infty + i\varphi_1$ and terminating at the point $+\infty + i\varphi_2$ with $-\infty < \varphi_1 < \varphi_2 < +\infty$.

(iii) $\Omega = L_{i\gamma\infty}$ is a contour starting at the point $\gamma - i\infty$ and terminating at the point $\gamma + i\infty$, where $\gamma \in R = (-\infty, +\infty)$.

A detailed and comprehensive account of the H-function is available from the monograph by Mathai and Saxena (1978), Prudnikov et al. (1989), and Kilbas and Saigo (2004). The relation connecting the ${}_p\Psi_q(z)$ and the H-function is given for the first time in the monograph by Mathai and Saxena

(1978, p.11, Eq.1.7.8) as

$${}_p\Psi_q[(a_1, A_1), \dots, (a_p, A_p) | z] = H_{p, q+1}^{1, p}[-z \mid_{(0, 1), (1-b_1, B_1), \dots, (1-b_q, B_q)}^{(1-a_1, A_1), \dots, (1-a_p, A_p)}], \quad (8)$$

where ${}_p\Psi_q(z)$ is Wright's generalized hypergeometric function, (Wright, (1935, 1940); also see Erdélyi, Magnus, Oberhettinger, and Tricomi (1953, Section 4.1), defined by means of the series representation in the form

$${}_p\Psi_q(z) = {}_p\Psi_q[(a_p, A_p) | z] = \sum_{r=0}^{\infty} \frac{[\prod_{j=1}^p \Gamma(a_j + A_j r)] z^r}{[\prod_{j=1}^q \Gamma(b_j + B_j r)] (r)!}, \quad (9)$$

where $z \in C$, $a_i, b_j \in C$, $A_i, B_j \in R_+$; $i = 1, \dots, p$; $j = 1, \dots, q$, $\sum_{j=1}^q b_j - \sum_{j=1}^p A_j > -1$; C being the complex number field. This function includes many special functions besides the Mittag-Leffler function defined by the equations (3) and (4). It is interesting to observe that for $A_i = B_j = 1 \ \forall \ i$ and j , (9) reduces to a generalized hypergeometric function ${}_pF_q(z)$. Thus

$${}_p\Psi_q[(a_p, 1) | z] = \frac{\prod_{j=1}^p \Gamma(a_j)}{\prod_{j=1}^q \Gamma(b_j)} {}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; z), \quad (10)$$

where $a_j \neq -\nu$ ($j = 1, \dots, p$ and $\nu = 0, 1, 2, \dots$); $p < q$ (or $p = q, |z| < 1$).

Prior to (9), Wright (1933) introduced a special case of (9) in the form

$$\Phi(a, b; z) = {}_0\Psi_1[\bar{-(b, a)} | z] = \sum_{r=0}^{\infty} \frac{1}{\Gamma(ar + b)} \frac{z^r}{(r)!},$$

which widely occurs in problems of fractional diffusion. It has been shown by Saxena, Mathai, and Haubold (2004b) that

$$E_{\alpha, \beta}(z) = {}_1\Psi_1[\bar{-(1, \alpha)} | z] \quad (11)$$

$$= H_{1, 2}^{1, 1}[-z \mid_{(0, 1), (1-\beta, \alpha)}^{(0, 1)}]. \quad (12)$$

If we further take $\beta = 1$ in (11) and (12), we find that

$$E_{\alpha, 1}(z) = E_{\alpha}(z) = {}_1\Psi_1[\bar{-(1, \alpha)} | z] \quad (13)$$

$$= H_{1, 2}^{1, 1}[-z \mid_{(0, 1), (0, \alpha)}^{(0, 1)}], \quad (14)$$

where $Re(\alpha) > 0, \alpha \in C$. The definitions of the well-known Laplace and Fourier transforms of a function $N(x, t)$ and their inverses are described below.

The Laplace transform of a function $N(x, t)$ with respect to t is defined by

$$L \{N(x, t)\} = \int_0^\infty e^{-st} N(x, t) dt, \quad t > 0, x \in R, \quad (15)$$

where $Re(s) > 0$, and its inverse transform with respect to s is given by

$$L^{-1} \{N(x, t)\} = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} N(x, s) ds, \quad (16)$$

γ being a fixed real number.

The Fourier transform of a function $N(x, t)$ with respect to x is defined by

$$F \{N(x, t)\} = \int_{-\infty}^\infty e^{ikx} N(k, t) dk. \quad (17)$$

The inverse Fourier transform with respect to k is given by the eq.

$$F^{-1} \{N(x, t)\} = \frac{1}{2\pi} \int_{-\infty}^\infty e^{-ikx} N(k, t) dk. \quad (18)$$

The space of functions for which the transforms defined by (15) and (17) exist is denoted by $LF = L(R_+) \times F(R)$.

In view of the results of Saxena, Mathai, and Haubold (2004a, p.49), also see Prudnikov, Brychkov, and Marichev (1989, p.355, Eq.2.25.3.2), the cosine transform of the H-function is given by

$$\begin{aligned} & \int_0^m t^{\rho-1} \cos(kt) H_{p,q}^{m,n} [at^\mu \mid (a_p, a_p)(b_q, B_q)] dt \\ &= \frac{\pi}{k^\rho} H_{q+1, p+2}^{n+1, m} \left[\frac{k^\mu}{a} \mid (1-b_q, B_q), (\frac{1+\rho}{2}, \frac{\mu}{2}) \right. \\ & \quad \left. (\rho, \mu), (1-a_p, A_p), (\frac{1+\rho}{2}, \frac{\mu}{2}) \right], \end{aligned} \quad (19)$$

where $Re[\rho + \mu \min_{1 \leq j \leq m} (\frac{b_j}{B_j})] > 1, |arg a| < \frac{1}{2}\pi\theta; \theta = \sum_{j=1}^n A_j - \sum_{j=n+1}^p A_j + \sum_{j=1}^m B_j - \sum_{j=m+1}^q B_j > 0$.

The Riemann-Liouville fractional integral of order ν is defined by (Miller and Ross, 1993, p. 45)

$${}_0D_t^{-\nu} N(x, t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-u)^{\nu-1} N(x, u) du, \quad (20)$$

where $Re(\nu) > 0$.

Following Samko, Kilbas, and Marichev (1990, p. 37), we define the fractional derivative of order $\alpha > 0$ in the form

$${}_0D_t^\alpha N(x, t) = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_0^t \frac{N(x, u) du}{(t - u)^{\alpha - n + 1}}; \quad t > 0, (n = [\alpha] + 1, \quad (21)$$

where $[\alpha]$ means the integral part of the number α .

A comprehensive account of the various applications of fractional calculus in physics is available from the monograph of Frank (2005).

From Erdélyi et al. (1954a and 1954b, p.182), we have

$$L \{ {}_0D_t^{-\nu} N(x, t) \} = s^{-\nu} F(x, s), \quad (22)$$

where $F(x, s)$ is the Laplace transform with respect to t of $N(x, t)$, $Re(s) > 0$ and $Re(\nu) > 0$.

The Laplace transform of the fractional derivative, defined by (21), is given by Oldham and Spanier (1974, p. 134, Eq.(8.1.3))

$$L \{ {}_0D_t^\alpha N(x, t) \} = s^\alpha N(x, s) - \sum_{r=1}^n s^{r-1} {}_0D_t^{\alpha-r} N(x, t)|_{t=0}, \quad n-1 < \alpha \leq n. \quad (23)$$

In certain boundary value problems the following fractional derivative of order α is introduced by Caputo (1969) in the form

$${}_0D_t^\alpha f(x, t) = \frac{1}{\Gamma(m - \alpha)} \int_0^t \frac{f^{(m)}(x, \tau) d\tau}{(t - \tau)^{\alpha + 1 - m}} \quad (24)$$

$$\begin{aligned} m - 1 < \alpha \leq m, Re(\alpha) > 0, m \in N, \\ &= \frac{\partial^m f(x, t)}{\partial t^m}, \text{ if } \alpha = m, \end{aligned} \quad (25)$$

where $\frac{d^m}{dt^m} f$ is the m^{th} derivative of order m of the function $f(x, t)$ with respect to t . The Laplace transform of this derivative is given by Podlubny (1999) in the form

$$L \{ {}_0D_t^\alpha f(x, t); s \} = s^\alpha F(x, s) - \sum_{r=0}^{m-1} s^{\alpha-r-1} f^{(r)}(x, 0+), \quad m-1 < \alpha \leq m. \quad (26)$$

The above formula is useful in deriving the solution of differential and integral equations of fractional order governing certain physical problems of reaction and diffusion .

We also need the Weyl fractional derivative, defined by

$${}_{-\infty}D_x^\mu f(x, t) = \frac{1}{\Gamma(n - \mu)} \frac{d^n}{dx^n} \int_{-\infty}^x \frac{f(u, t) du}{(x - u)^{\mu - n + 1}}, \quad (27)$$

where $x \in R, \mu > 0, n = [\mu] + 1, [\mu]$ being the integer part of $\mu > 0$ (Samko et al, 1990, Section 24.2).

Its Fourier transform is given by (Metzler and Klafter, 2000, p.59, A.11)

$$F \{ {}_{-\infty}D_x^\mu f(x, t) \} = (ik)^\mu \Psi(k, t), \mu > 0, \quad (28)$$

where $\Psi(k, t)$ is the Fourier transform of $f(x, t)$ with respect to the variable x of $f(x, t)$. Following the convention initiated by Compte (1996), we suppress the imaginary unit in Fourier space by adopting the slightly modified form of above result in our investigations (Metzler and Klafter, 2000, p.59, A.12)

$$F \{ {}_{-\infty}D_x^\mu f(x, t) \} = -|k|^\mu \Psi(k, t) \quad (29)$$

instead of (28). Finally we also need the following property of the H-function (Mathai and Saxena 1978)

$$H_{p,q}^{m,n} [x^\delta \mid_{(b_q, B_q)}^{(a_p, A_p)}] = \frac{1}{\delta} H_{p,q}^{m,n} [x \mid_{(b_q, B_q/\delta)}^{(a_p, A_p/\delta)}], \quad (30)$$

where $\delta > 0$.

3 Fractional Reaction-Diffusion Equation

In this section, we will investigate the solution of the generalized reaction-diffusion equation (31). The result is given in the form of the following

Theorem. Consider the generalized fractional-reaction diffusion model

$${}_0D_t^\beta N(x, t) = \eta {}_{-\infty}D_x^\alpha N(x, t) + \varphi(x, t); \eta > 0, t > 0, x \in R, 1 < \beta \leq 2, \quad (31)$$

$0 \leq \alpha \leq 1$ with the initial conditions

$${}_0D_t^{\alpha-1} N(x, 0) = f(x), {}_0D_t^{\alpha-2} N(x, 0) = g(x), x \in R, \lim_{x \rightarrow \pm\infty} N(x, t) = 0, \quad (32)$$

where ${}_0D_t^{\alpha-1} N(x, 0)$ means the Riemann-Liouville fractional derivative of order $\alpha - 1$ with respect to t evaluated at $t = 0$. Similarly, ${}_0D_t^{\alpha-2} N(x, 0)$

means the Riemann-Liouville fractional derivative of order $\alpha - 2$ with respect to t evaluated at $t = 0$. The quantity η is a diffusion constant and $\varphi(x, t)$ is a nonlinear function belonging to the area of reaction-diffusion. Then for the solution of (31), subject to the initial conditions (32), there holds the eq.

$$\begin{aligned} N(x, t) &= \frac{t^{\beta-1}}{2\pi} \int_{-\infty}^{\infty} \Psi(k) E_{\beta,\beta}(-\eta|k|^{\alpha}t^{\beta}) \exp(-ikx) dk \\ &+ \frac{t^{\beta-2}}{2\pi} \int_{-\infty}^{\infty} \tilde{g}(k) E_{\beta,\beta-1}(-\eta|k|^{\alpha}t^{\beta}) \exp(-ikx) dk \\ &+ \frac{1}{2\pi} \int_0^{\infty} \xi^{\beta-1} \int_{-\infty}^{\infty} \tilde{\varphi}(k, t - \xi) E_{\beta,\beta}(-\eta|k|^{\alpha}\xi^{\beta}) \exp(-ikx) dk d\xi, \end{aligned} \quad (33)$$

where \sim indicates the Fourier transform with respect to the space variable x .

Proof. If we apply the Laplace transform with respect to the time variable t and use eq. (23), eq. (31) becomes

$$s^{\beta} N^{*}(x, s) - f(x) - sg(x) = \eta_{-\infty} D_x^{\alpha} N^{*}(x, s) + \varphi^{*}(x, s).$$

As is customary, it is convenient to employ the symbol $*$ to indicate the Laplace transform with respect to the variable t .

Now we apply the Fourier transform with respect to the space variable x to the above equation and use the initial conditions and the result (29), then the above equation transforms into the form

$$\tilde{N}^{*}(k, s) = \frac{\tilde{f}(k)}{s^{\beta} + \eta|k|^{\alpha}} + \frac{\tilde{sg}(k)}{s^{\beta} + \eta|k|^{\alpha}} + \frac{\tilde{\varphi}^{*}(k)}{s^{\beta} + \eta|k|^{\alpha}}, \quad (34)$$

where, according to convention followed, \sim indicates the Fourier transform with respect to x . On taking the inverse Laplace transform of (34) and using the result

$$L^{-1} \left\{ \frac{s^{\beta-1}}{a + s^{\alpha}}; t \right\} = t^{\alpha-\beta} E_{\alpha, \alpha-\beta+1}(-at^{\alpha}), \quad (35)$$

where $Re(s) > 0, Re(\alpha - \beta) > -1$, it is seen that

$$\begin{aligned} \tilde{N}(k, t) &= \tilde{f}(k) t^{\beta-1} E_{\beta,\beta}(-\eta|k|^{\alpha}t^{\beta}) + \tilde{g}(k) t^{\beta-2} E_{\beta,\beta-1}(-\eta|k|^{\alpha}t^{\beta}) \\ &+ \int_0^{\infty} \tilde{\varphi}(k, t - \xi) \xi^{\beta-1} E_{\beta,\beta}(-\eta|k|^{\alpha}\xi^{\beta}) d\xi. \end{aligned} \quad (36)$$

The required solution (33) now readily follows by taking the inverse Fourier transform of (36). Thus, we obtain

$$\begin{aligned} N(x, t) &= \frac{t^{\beta-1}}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(k) E_{\beta,\beta}(-\eta|k|^{\alpha}t^{\beta}) \exp(-ikx) dk \\ &+ \frac{t^{\beta-2}}{2\pi} \int_{-\infty}^{\infty} \tilde{g}(k) E_{\beta,\beta-1}(-\eta|k|^{\alpha}t^{\beta}) \exp(-ikx) dk \\ &+ \frac{1}{2\pi} \int_0^t \xi^{\beta-1} \int_{-\infty}^{\infty} \varphi^*(k, t-\xi) E_{\beta,\beta}(-\eta|k|^{\alpha}\xi^{\beta}) \exp(-ikx) dk d\xi. \end{aligned}$$

This completes the proof of the theorem.

Note 1. It may be noted here that by virtue of the identity (12), the solution (33) can be expressed in terms of the H-function as can be seen from the solutions given in the special cases of the theorem in the next section. Further we observe that (33) is not an explicit solution and special cases are of interest.

4 Special Cases

When $g(x) = 0$, then applying the convolution theorem of the Fourier transform to the solution (33), the theorem yields

Corollary 1.1. The solution of fractional reaction-diffusion equation

$${}_0D_t^{\beta} N(x, t) = \eta {}_{-\infty}D_x^{\alpha} N(x, t) + \varphi(x, t), t > 0, \eta > 0, \quad (37)$$

subject to the initial conditions,

$${}_0D_t^{\alpha-1} N(x, t)|_{t=0} = f(x), {}_0D_t^{\alpha-2} N(x, t)|_{t=0} = 0 \quad (38)$$

$$\text{for } x \in R, \lim_{x \rightarrow \pm\infty} N(x, t) = 0, 1 < \beta \leq 2,$$

$0 \neq \alpha \neq 1$, where η is a diffusion, constant and $\varphi(x, t)$ is a nonlinear function belonging to the area of reaction diffusion is given by

$$\begin{aligned} N(x, t) &= \int_{-\infty}^{\infty} G_1(x - \tau, t) f(\tau) d\tau \\ &+ \int_0^t (t - \xi)^{\beta-1} \int_0^x G_2(x - \tau, t - \xi) \varphi(\tau, \xi) d\tau d\xi, \end{aligned} \quad (39)$$

where

$$\begin{aligned}
G_1(x, t) &= \frac{t^{\beta-1}}{2\pi} \int_{-\infty}^{\infty} \exp(-ikx) E_{\beta, \beta}(-\eta |k|^{\alpha} t^{\beta}) dk \\
&= \frac{t^{\beta-1}}{\pi \alpha} \int_0^{\infty} \cos(kx) H_{1,2}^{1,1} [k \eta^{1/\alpha} t^{\beta/\alpha} \mid_{(0,1/\alpha), (1-\beta, \beta/\alpha)}^{(0,1/\alpha)}] dk \\
&= \frac{t^{\beta-1}}{\alpha |x|} H_{3,3}^{2,1} \left[\frac{|x|}{\eta^{1/\alpha} t^{\beta/\alpha}} \mid_{(1,1), (1,1/\alpha), (1,1/2)}^{(1,1/\alpha), (\beta, \beta/\alpha), (1,1/2)} \right], \quad \text{Re}(\alpha) > 0, \quad (40)
\end{aligned}$$

$$\begin{aligned}
G_2(x, t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-ikx) E_{\beta, \beta}(-\eta |k|^{\alpha} t^{\beta}) dk \\
&= \frac{1}{\pi \alpha} \int_0^{\infty} \cos(kx) H_{1,2}^{1,1} [k \eta^{1/\alpha} t^{\beta/\alpha} \mid_{(0,1/\alpha), (1-\beta, \beta/\alpha)}^{(0,1/\alpha)}] dk \\
&= \frac{1}{\alpha |x|} H_{3,3}^{2,1} \left[\frac{|x|}{\eta^{1/\alpha} t^{\beta/\alpha}} \mid_{(1,1), (1,1/\alpha), (1,1/2)}^{(1,1/\alpha), (\beta, \beta/\alpha), (1,1/2)} \right], \quad \text{Re}(\alpha) > 0. \quad (41)
\end{aligned}$$

If we set $f(x) = \delta(x)$, where $\delta(x)$ is the Dirac delta function, then we arrive at

Corollary 1.2. Consider the following reaction-diffusion model

$$\frac{d^{\beta} N(x, t)}{dt^{\beta}} = \eta {}_{-\infty} D_x^{\alpha} N(x, t), \quad \eta > 0, \quad x \in R, \quad (42)$$

with the initial condition $N(x, t = 0) = \delta(x)$, $\lim_{x \rightarrow \pm\infty} N(x, t) = 0$, $0 < \beta \leq 1$, where η is a diffusion constant and $\delta(x)$ is the Dirac delta function. Then the solution of (42) under the given initial conditions is given by

$$N(x, t) = \frac{t^{\beta-1}}{\alpha |x|} H_{3,3}^{2,1} \left[\frac{|x|}{(\eta t^{\beta})^{1/\alpha}} \mid_{(1,1), (1,1/\alpha), (1,1/2)}^{(1,1/\alpha), (\beta, \beta/\alpha), (1,1/2)} \right], \quad (43)$$

where $\text{Re}(\alpha) > 0$, $\text{Re}(\beta) > 2$.

When $\beta = 1/2$, the above corollary reduces to the following result.

Consider the following reaction-diffusion model

$$\frac{d^{1/2} N(x, t)}{dt^{1/2}} = \eta {}_{-\infty} D_x^{\alpha} N(x, t), \quad \eta > 0, \quad x \in R, \quad (44)$$

with the initial condition $N(x, t = 0) = \delta(x)$, $\lim_{x \rightarrow \pm\infty} N(x, t) = 0$, where η is a diffusion constant and $\delta(x)$ is the Dirac delta function. Then the solution

of (42) under the given initial conditions is given by

$$N(x, t) = \frac{1}{\alpha |x| t^{1/2}} H_{3,3}^{2,1} \left[\frac{|x|}{(\eta t^{1/2})^{1/\alpha}} \begin{matrix} (1,1/\alpha), (1/2, 1/2\alpha), (1,1/2) \\ (1,1), (1,1/\alpha), (1,1/2) \end{matrix} \right],$$

where $Re(\alpha) > 0, Re(\beta) > 0$.

Remark 1. The solution of eq. (42), as given by Kilbas et al (2005) is in terms of the inverse Laplace and inverse Fourier transforms of certain functions whereas our solution of the same equation is obtained in an explicit closed form in terms of the H-function.

An interesting case is, when $\beta \rightarrow 1$, then in view of the cancellation law for the H-function (Mathai and Saxena, 1978), (43) provides the following result given by Jespersen et al. (1999) and recently by del-Castillo-Negrete et al. (2003) in an entirely different form.

For the solution of fractional-reaction-diffusion equation

$$\frac{d}{dt} N(x, t) = \eta_{-\infty} D_x^\alpha N(x, t), \quad (45)$$

with initial condition

$$N(x, t = 0) = \delta(x), \quad \lim_{x \rightarrow \pm\infty} N(x, t) = 0,$$

there holds the relation

$$N(x, t) = \frac{1}{\alpha |x|} H_{2,2}^{1,1} \left[\frac{|x|}{\eta^{1/\alpha} t^{1/\alpha}} \begin{matrix} (1,1/\alpha), (1,1/2) \\ (1,1), (1,1/2) \end{matrix} \right], \quad (46)$$

where $Re(\alpha) > 0$. In passing, it may be noted that (46) is a closed form representation of a Lévy stable law, see Metzler and Klafter (2000). It is interesting to note that as $\alpha \rightarrow 2$, the classical Gaussian solution is recovered as

$$\begin{aligned} N(x, t) &= \frac{1}{2|x|} H_{2,2}^{1,1} \left[\frac{|x|}{(\eta t)^{1/2}} \begin{matrix} (1,1/2), (1,1/2) \\ (1,1), (1,1/2) \end{matrix} \right] \\ &= \frac{1}{2|x|} H_{1,1}^{1,0} \left[\frac{|x|}{(\eta t)^{1/2}} \begin{matrix} (1,1/2) \\ (1,1) \end{matrix} \right] \\ &= (4\pi\eta t)^{-1/2} \exp\left(-\frac{|x|^2}{4\eta t}\right). \end{aligned} \quad (47)$$

It is useful to study the solution (43) due to its occurrence in certain fractional diffusion models. we will find the fractional order moments of (43) in the next section.

Remark 2. Applying Fourier transform with respect to x in (42), it is found that

$$\frac{d^\beta}{dt^\beta} \Psi(k, t) = -\eta |k|^\alpha \Psi(k, t), \quad 0 < \beta \leq 1, \quad (48)$$

which is the generalized Fourier transformed diffusion equation, since for $\alpha = 2$ and for $\beta \rightarrow 1$, it reduces to the Fourier transformed diffusion equation

$$\frac{d\Psi(k, t)}{dt} = -\eta |k|^2 \Psi(k, t),$$

being a relaxation (diffusion) equation, for a fixed wave number k (Metzler and Klafter, 2000). Here $\Psi(k, t)$ is the Fourier transform of $N(x, t)$ with respect to x .

Remark 3. It is interesting to observe that the method employed for deriving the solution of the equations (31) and (32) in the space $= LF = L(R_+) \times F(R)$ can also be applied in the space $LF' = L'(R_+) \times F'$, where $F' = F'(r)$ is the space of Fourier transform of generalized functions. As an illustration, we can choose $F' = S'$ or $F' = D'$. The Fourier transforms in S' and D' are introduced by Gelfand and Shilov (1964). S' is the dual of the space S , which is the space of all infinitely differentiable functions which, together with their derivatives approach zero more rapidly than any power of $1/|x|$ as $|x| \rightarrow \infty$ (Gelfand and Shilov, 1964, p.16). D' is the dual of the space D which consists of all smooth functions with compact supports (Brychkov and Prudnikov, 1989 p. 3). For further details, the reader is referred to the monographs by Gelfand and Shilov (1964) and Brychkov and Prudnikov (1989), if we replace the Laplace and Fourier transforms in (15) and (17) by the corresponding Laplace and Fourier transform of generalized functions.

5 Fractional Order Moments

In this section, we will calculate the fractional order moments, defined by

$$\langle |x|^\delta \rangle = \int_{-\infty}^{\infty} |x|^\delta N(x, t) dx. \quad (49)$$

Using the definition of the Mellin transform

$$M \{f(t); s\} = \int_0^\infty t^{s-1} f(t) dt, \quad (50)$$

we find from (49) that

$$\langle |x(t)|^\delta \rangle = \int_{-\infty}^\infty |x|^\delta N(x, t) dx \quad (51)$$

$$\langle |x|^\delta(t) \rangle = \frac{2t^{\beta-1}}{\alpha} \int_0^\infty x^{\delta-1} H_{3,3}^{2,1} \left[\frac{|x|}{\eta^{1/\alpha} t^{\beta/\alpha}} \middle| \begin{matrix} (1, 1/\alpha), (1, \beta/\alpha), (1, 1/2) \\ (1, 1), (1, 1/\alpha), (1, 1/2) \end{matrix} \right] dx. \quad (52)$$

Applying the Mellin transform formula for the H-function, namely

$$\int_0^\infty x^{\rho-1} H_{p,q}^{m,n} [ax \mid \begin{matrix} (a_p, A_p) \\ (b_q, b_q) \end{matrix}] dx = a^{-\rho} \Theta(-\rho), \quad (53)$$

where

$$\min_{1 \leq j \leq m} \operatorname{Re} \left(\frac{b_j}{B_j} \right) < \operatorname{Re}(\rho) < \max_{1 \leq j \leq n} \operatorname{Re} \left(\frac{1 - a_j}{A_j} \right), \quad |\arg a| < \frac{1}{2} \pi \theta, \theta > 0,$$

Θ is defined in (19) and $\Theta(-\rho)$ in the definition of the H-function (5), we see that

$$\langle |x|^\delta(t) \rangle = \frac{2}{\alpha} \eta^{\delta/\alpha} t^{\beta\{(\delta/\alpha)+1-(1/\beta)\}} \frac{\Gamma(-\frac{\delta}{\alpha}) \Gamma(1+\delta) \Gamma(1+\frac{\delta}{\alpha})}{\Gamma(-\frac{\delta}{2}) \Gamma(\beta + \frac{\beta\delta}{\alpha}) \Gamma(1+\frac{\delta}{2})}, \quad (54)$$

whenever the gammas exist, $\operatorname{Re}(\delta) > -1$ and $\operatorname{Re}(\delta + \alpha) > 0$.

Two interesting special cases of (54) are worth mentioning.

$$(i) \text{ As } \delta \rightarrow 0, \text{ then using the result } \frac{1}{\Gamma(z)} \sim z \text{ for } z \ll 1, \quad (55)$$

we find that

$$\lim_{\delta \rightarrow 0} \langle |x|^\delta(t) \rangle = \frac{t^{\beta-1}}{\Gamma(\beta)}. \quad (56)$$

(ii) When $\alpha = 2, \delta = 2$, the linear time dependence,

$$\lim_{\delta \rightarrow 2, \alpha \rightarrow 2} \langle |x(t)|^\delta \rangle = \frac{2\eta t^{2\beta-1}}{\Gamma(2\beta)}, \quad (57)$$

of the mean squared displacement is recovered.

6 Behavior of the Solution in Equation (43)

Eq. (43) can be expressed in terms of the Mellin-Barnes type integral (Erdélyi et al, 1953, chapter 1) as

$$N(x, t) = \frac{t^{\beta-1}}{\alpha|x|} \frac{1}{2\pi} \int_L \frac{\Gamma(\frac{s}{\alpha})\Gamma(1-s)\Gamma(1-\frac{s}{\alpha})}{\Gamma(\beta-\frac{s\beta}{\alpha})\Gamma(1-\frac{s}{2})\Gamma(s/2)} \left[\frac{|x|}{\eta^{1/\alpha}t^{\beta/\alpha}}\right]^s ds. \quad (58)$$

Let us assume that the poles of the integrand of (58) are simple. Now evaluating the sum of residues in ascending powers of $|x|$ by calculating the residues at the poles of $\Gamma(1-s)$ at the points $s = 1 + \nu$ ($\nu = 0, 1, 2, \dots$), and $\Gamma(1-\frac{s}{\alpha})$ at the points $s = 1 + \nu$ ($\nu = 0, 1, 2, \dots$), it is found that the series expansion of the general solution (43) is given by

$$\begin{aligned} N(x, t) &= \frac{t^{\beta-1}}{\alpha\eta^{1/\alpha}t^{\beta/\alpha}} \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{(\nu)!} \frac{\Gamma[\frac{1+\nu}{\alpha}]\Gamma(1-(1+\nu)/\alpha)}{\Gamma[1-\frac{\beta(1+\nu)}{\alpha}]\Gamma[\frac{1-\nu}{2}]\Gamma[\frac{1+\nu}{2}]} \left[\frac{|x|}{\eta^{1/\alpha}t^{\beta/\alpha}}\right]^\nu \\ &+ \frac{|x|^{\alpha-1}}{\eta t} \sum_{\nu=0}^{\infty} \frac{(-1)^\nu \Gamma(1-\alpha(1+\nu))}{\Gamma(-\beta\nu)\Gamma(1-\frac{\alpha}{2}(1+\nu))\Gamma(\frac{\alpha(1+\nu)}{2})} \left[\frac{|x|^\alpha}{(\eta t^\beta)^{1/\alpha}}\right]^{\alpha\nu} \end{aligned} \quad (59)$$

where $0 < Re(\nu) < \alpha, \left\{\frac{|x|}{\eta^{1/\alpha}t^{1/\alpha}}\right\} < 1$.

From (59), we infer that

$$N(x, t) \sim At^{\beta-\beta/\alpha-1} + B|x|^{\alpha-1}, \text{ as } x \rightarrow 0, \quad (60)$$

where A and B are numerical constants. Further from the series expansion (59), it can be seen that the initial behavior is given by

$$N(x, t) \sim \frac{\Gamma(1+\frac{1}{\alpha})\Gamma(1-\frac{1}{\alpha})}{\pi\eta^{1/\alpha}t^{1+\beta/\alpha-\beta}\Gamma(\beta-\frac{\beta}{\alpha})} \text{ for } 1 < \alpha < 2, \quad (61)$$

with $\left\{\frac{|x|}{(\eta t^\beta)^{1/\alpha}}\right\} < 1$.

Next, if we calculate the residues at the poles of $\Gamma(s/\alpha)$ at the points $s = -\alpha\nu$ ($\nu = 0, 1, 2, \dots$), it gives

$$N(x, t) = \frac{t^{\beta-1}}{|x|} \sum_{\nu=0}^{\infty} \frac{\Gamma(1+\alpha\nu)}{\Gamma(\beta+\beta\nu)\Gamma(1+\frac{\alpha\nu}{2})\Gamma(-\frac{\alpha\nu}{2})} \left[-\frac{\eta t^\beta}{|x|^\alpha}\right]^\nu, \quad (62)$$

for $|x| > 1$. From (62), it can be readily seen that

$$N(x, t) \sim \frac{t^{\beta-1}}{|x|}$$

for large $|x|$. In conclusion, it is observed that the solution given by (43) does not admit a probabilistic interpretation in contrast with fractional reaction-diffusion based on Caputo derivative derived by the authors. However, when $\beta \rightarrow 1$ then it has a probabilistic interpretation, as shown in corollary 1.2.

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